

# UNIQUENESS OF LEFT INVERSES IN CONVEX DOMAINS, SYMMETRIZED BIDISC AND TETRABLOCK

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ABSTRACT. In the paper we discuss the problem of uniqueness of left inverses (solutions of two point Nevanlinna-Pick problem) in bounded convex domains, strongly linearly convex domains, the symmetrized bidisc and the tetrablock.

## 1. MOTIVATION: GEODESICS VS. LEFT INVERSES

The problem we are dealing with has two origins. The one is connected with the equality of the Lempert function and the Carathéodory distance of two different points  $w, z$  in a taut domain  $D$  which is equivalent to the existence of holomorphic functions:  $f : \mathbb{D} \rightarrow D$  and  $F : D \rightarrow \mathbb{D}$  such that  $f$  passes through  $w$  and  $z$  and  $F \circ f = \text{id}_{\mathbb{D}}$ . We call such an  $f$  a *complex geodesic* and  $F$  a *left inverse (of  $f$ )*. The most general result in this direction is the Lempert Theorem on equality of the Lempert function and the Carathéodory distance in the class of strongly linearly convex domains (and simultaneously the uniqueness of the complex geodesics in that class of domains) or convex domains (in general without the uniqueness of geodesics). The other origin is the Nevanlinna-Pick problem which extensively studied in the case of the unit disc has also been recently investigated in higher dimensional domains, especially in the polydisc. In our paper we study the Nevanlinna-Pick problem for two points in more general higher dimensional (of dimension at least two) domains.

To illustrate the problem let us present two examples: the Euclidean (two-dimensional) ball and the bidisc. Complex geodesics in the Euclidean ball are uniquely determined whereas in the bidisc they are (generically) non-unique. It is natural that we may study the problem of uniqueness of left inverses. In the case of the Euclidean ball the left inverses are non-uniquely determined. Actually, for the complex geodesic  $\mathbb{D} \ni \lambda \mapsto (\lambda, 0) \in \mathbb{B}_2$  the functions  $\mathbb{B}_2 \ni z \mapsto \frac{z_1}{\sqrt{1+\gamma z_2^2}} \in \mathbb{D}$ ,

where  $\gamma \in \overline{\mathbb{D}}$  are left inverses. On the other hand the non-uniquely determined geodesics for points  $(0, 0)$  and  $(\lambda, \gamma\lambda)$  in the bidisc, where  $|\gamma| < 1$ , determine uniquely the left inverse ( $\mathbb{D}^2 \ni z \mapsto z_1 \in \mathbb{D}$ ). But the points  $(0, 0)$  and  $(\lambda, \gamma\lambda)$ ,  $|\gamma| = 1$  (with uniquely determined geodesics) have many left inverses: for instance,  $\mathbb{D}^2 \ni z \mapsto tz_1 + (1-t)\bar{\gamma}z_2 \in \mathbb{D}$ ,  $t \in [0, 1]$ . Here we see a correspondence: uniqueness of geodesics corresponds to non-uniqueness of left inverses. This may be rephrased as follows: a complex geodesic  $f : \mathbb{D} \rightarrow \mathbb{D}^2$  has a non-uniquely determined left

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2010 *Mathematics Subject Classification.* Primary 32F17, 32F45; Secondary 47A57.

*Key words and phrases.* complex geodesic, left inverse, Nevanlinna-Pick problem, strongly linearly convex domain, symmetrized bidisc, tetrablock.

The work is partially supported by the grant of the Polish National Science Centre no. UMO-2011/03/B/ST1/04758.

inverse iff both  $f_1$  and  $f_2$  are automorphisms of the unit disc (this is actually mentioned in [Agl-McC 2003] while introducing a notion of a balanced pair). This is the problem that is a starting one in our paper.

In most cases studied by us there is such a correspondence. More precisely, we show that in the class of strongly linearly convex domains any complex geodesic has many left inverses (see Theorem 3.1) whereas in the case of convex domains the non-uniqueness of geodesics implies the uniqueness of left inverses (this is exactly the case in two dimensional case, in higher dimension there must be many more geodesics - see Theorem 4.1). Recall that strongly linearly convex domains admit uniquely determined geodesics. There are, however, other convex domains with uniquely determined complex geodesics and yet determining uniquely left inverses (see Proposition 4.4).

The above discussion and the results presented show that the situation in the case of convex domains is relatively well understood and to some extent regular. We may go, however, beyond that class of domains. Recall that there are two  $\mathbb{C}$ -convex domains not biholomorphic to convex ones for which the Lempert Theorem holds (i.e. the Carathéodory function and the Lempert function coincide). These domains although somehow related turn out to be quite different as the uniqueness of the geodesics is concerned. The symmetrized bidisc has uniquely determined geodesics and yet many of them (but not all!) admit unique left inverses (see Theorem 5.3) whereas the tetrablock has generically non-uniqueness of geodesics but the problem of uniqueness of left inverses is a much more complex one (see Section 6). We could settle down the problem but the philosophy which is behind that phenomenon is still a mystery to us.

Note that the study of the problem of uniqueness of left inverses in the symmetrized bidisc reduces, in some cases, to the uniqueness of the Nevanlinna-Pick problem for three points in the bidisc. Moreover, in our study in these two domains we rely very much upon the description of holomorphic retracts in the bidisc (see Theorem 5.1).

Although the results presented in the paper allow us to understand the phenomenon of uniqueness of left inverses the form of left inverses in the non-uniqueness case is not understood at all. The authors have a very vague idea that a notion of a magic function (see [Agl-You 2008]) could be a useful tool in handling that problem.

## 2. DEFINITIONS, NOTATIONS AND KNOWN RESULTS

To start with let us recall basic definitions, notation, facts and results that we shall use.

For a domain  $D \subset \mathbb{C}^n$ ,  $w, z \in D$  we define two holomorphically invariant functions. *The Lempert function* is defined as follows

$$(2.1) \quad \tilde{k}_D(w, z) := \inf \{p(\lambda_1, \lambda_2)\},$$

where the infimum is taken over all  $\lambda_1, \lambda_2 \in \mathbb{D}$  and holomorphic mappings  $f : \mathbb{D} \rightarrow D$  such that  $f(\lambda_1) = w$ ,  $f(\lambda_2) = z$ . Here  $p$  denotes the Poincaré distance on the unit disc  $\mathbb{D}$ .

*The Carathéodory pseudodistance* is defined by the formula

$$(2.2) \quad c_D(w, z) := \sup \{p(F(w), F(z))\},$$

where the supremum is taken over all holomorphic functions  $F : D \rightarrow \mathbb{D}$ . Recall that  $c_D \leq \tilde{k}_D$ . The equality  $\tilde{k}_D(w, z) = c_D(w, z)$  for some  $w, z \in D$ ,  $w \neq z$  is closely related to the existence of *complex geodesics*, i.e. holomorphic functions  $f : \mathbb{D} \rightarrow D$  such that  $w = f(\lambda_1^0)$ ,  $z = f(\lambda_2^0)$  and  $c_D(w, z) = p(\lambda_1^0, \lambda_2^0)$ . Recall that if  $f$  is a complex geodesic then so is  $f \circ a$  for any  $a \in \text{Aut}(\mathbb{D})$ , moreover in this case  $c_D(f(\lambda_1), f(\lambda_2)) = \tilde{k}_D(f(\lambda_1), f(\lambda_2)) = p(\lambda_1, \lambda_2)$  for any  $\lambda_1, \lambda_2 \in \mathbb{D}$ . Recall that in the case of  $D$  being taut for any pair of different points  $w, z \in D$  there is a function for which the infimum in the definition of the Lempert function is attained. In our paper we shall consider only such domains (more precisely, bounded convex, strongly linearly convex or  $\mathbb{C}$ -convex).

To present the basic result on the equality of the Lempert function and the Carathéodory pseudodistance recall that the smoothly bounded domain  $D \subset \mathbb{C}^n$  is called *strongly linearly convex* if there is a smooth defining function  $r$  of the domain  $D$  such that

$$(2.3) \quad \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(z) X_j \bar{X}_k > \left| \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial z_k}(z) X_j X_k \right|$$

for any  $z \in \partial D$  and any non-zero vector  $X$  from the complex tangent space to  $\partial D$  at  $z$ .

**Theorem 2.1.** (see [Lem 1981], [Lem 1984]) *Let  $D$  be a bounded convex domain in  $\mathbb{C}^n$ ,  $n \geq 1$  or let  $D$  be a strongly linearly convex domain in  $\mathbb{C}^n$ ,  $n > 1$ . Then  $\tilde{k}_D = c_D$ .*

*Moreover, for any points  $w, z \in D$ ,  $w \neq z$  there is a complex geodesic passing through  $w$  and  $z$ . In the case of strongly linearly convex domain the geodesics passing through a given pair of points are unique (up to an automorphism of  $\mathbb{D}$ ).*

In the case when  $f : \mathbb{D} \rightarrow D$  is a complex geodesic there is a holomorphic function  $F : D \rightarrow \mathbb{D}$  such that  $F \circ f$  is the identity of the unit disc. Note also that in the class of taut domains the fact that a holomorphic mapping  $f : \mathbb{D} \rightarrow D$  is a complex geodesic is actually equivalent to the existence of a holomorphic function  $F : D \rightarrow \mathbb{D}$  such that  $F \circ f = \text{id}_{\mathbb{D}}$ . In any case we call such an  $F$  *the left inverse to  $f$* .

For basic properties of the Lempert function and the Carathéodory distance and notions related to them we refer the Reader to [Jar-Pfl 1993].

Recall that recently two non-trivial domains which could not be directly handled by the Lempert theory turned out to satisfy the equality of the Lempert function and the Carathéodory distance. These are *the symmetrized bidisc*  $\mathbb{G}_2$  defined as the image of  $\mathbb{D}^2$  under the symmetrization mapping  $\mathbb{D}^2 \ni (\lambda_1, \lambda_2) \mapsto (\lambda_1 + \lambda_2, \lambda_1 \lambda_2)$  and *the tetrablock*  $\mathbb{E}$  defined as follows

$$(2.4) \quad \mathbb{E} := \{x \in \mathbb{C}^3 : |x_1 - \bar{x}_2 x_3| + |x_2 - \bar{x}_1 x_3| + |x_3|^2 < 1\}.$$

Recall that both domains are  $\mathbb{C}$ -convex (i.e. their intersection with any complex line is connected and simply connected), non-biholomorphically equivalent to convex domains and the Lempert function and Carathéodory distance coincide on them (see [Agl-You 2004], [Cos 2004], [Abo-Whi-You 2007], [Edi-Kos-Zwo 2012], [Nik-Pfl-Zwo 2008], [Zwo 2013]).

Let us fix a domain  $D \subset \mathbb{C}^n$ . Consider the following *Nevanlinna-Pick problem*: given  $N$  points (called *nodes*)  $z_1, \dots, z_N \in D$ , and  $N$  complex numbers  $\lambda_1, \dots, \lambda_N$

(called *targets*) decide whether there exists a holomorphic function  $F : D \rightarrow \mathbb{D}$  such that  $F(z_j) = \lambda_j$ . In case the function  $F$  is such that the supremum norm  $\|F\|$  is one and there is no solution of the Nevanlinna-Pick problem of the supremum norm smaller than one the solution  $F$  is called *extremal*. Note that in the case when  $N = 2$  and the equality  $\tilde{k}_D(w, z) = c_D(w, z) = p(\lambda_1, \lambda_2)$  (with holomorphic  $f : \mathbb{D} \rightarrow D$  such that  $f(\lambda_1) = w$ ,  $f(\lambda_2) = z$ ) holds, the (extremal) Nevanlinna-Pick problem for  $w, z$  and  $\lambda_1, \lambda_2$  reduces to finding the left inverse to  $f$ . In our paper we shall deal with this special Nevanlinna-Pick problem. We shall study the problem of uniqueness of the extremal solution in a wide class of domains for which the Lempert Theorem holds (bounded convex, strongly linearly convex domains, the symmetrized bidisc and the tetrablock). Among others we shall see that the two-point Nevanlinna-Pick problem in the symmetrized bidisc sometimes reduces to the three point Nevanlinna-Pick problem in the bidisc. There are many results on the Nevanlinna-Pick problem on the polydisc: among others the existence of the solutions, the uniqueness of the extremal problem and the structure of the set of uniqueness is studied (see e. g. [Agl-McC 2000], [Agl-McC 2002], [Bal-Tre 1998], [Guo-Hua-Wan 2008], [Sch 2011]).

Note that in the case  $N = 2$ , for  $D$  as above and for pairs  $(w, z)$  for which the Carathéodory and Lempert function coincide the Nevanlinna-Pick problem is actually the problem of finding the left inverses to a complex geodesic passing through  $w$  and  $z$ .

The results of the paper are the following. In Section 3 we show that the strongly linearly convex domains admit non-uniqueness of left inverses to all complex geodesics (Theorem 3.1). In Section 4 we provide a sufficient condition for uniqueness of left inverses under the assumption of existence of 'many' (in case  $n = 2$  two) left inverses passing through a given pair of points (Theorem 4.1) and we show that non-uniqueness of complex geodesics is not necessary for uniqueness of left inverses (see Proposition 4.4). In Sections 5 and 6 we discuss the problem of the uniqueness of left inverses in two examples of  $\mathbb{C}$ -convex domains for which the Lempert theory does not apply directly and thus the study of the uniqueness of complex geodesics and left inverses must be very specific. Both examples deliver unexpected phenomena and show its close connection with the Nevanlinna-Pick problem in the bidisc (see Theorem 5.3 and results of Section 6).

### 3. NON-UNIQUENESS OF LEFT INVERSES IN STRONGLY LINEARLY CONVEX DOMAINS

We start our study with presenting a result on non-uniqueness of left inverses for complex geodesics in the case of strongly linearly convex domains. The proof is based on a method of Lempert which enables us to reduce the problem to the same problem in the Euclidean unit ball.

**Theorem 3.1.** *Assume that  $D$  is smooth strongly linearly convex domain of  $\mathbb{C}^n$ ,  $n > 1$ . Then left inverses are never uniquely defined.*

*Proof.* Let  $f$  be a complex geodesic in  $D$ . It follows from [Lem 1984] (Proposition 11) that there exist a smooth domain  $G \subset \mathbb{C}^n$  and a biholomorphism  $\Phi : D \rightarrow G$  such that

- $\Phi(D) = G$ ;
- $g(\zeta) := \Phi(f(\zeta)) = (\zeta, 0, \dots, 0)$ ,  $\zeta \in \overline{\mathbb{D}}$ ;

- $\nu_G(g(\zeta)) = (\zeta, 0, \dots, 0)$ ,  $\zeta \in \partial\mathbb{D}$ , where  $\nu_G$  denotes the outer unit vector to  $\partial G$ ;
- for any  $\zeta \in \partial\mathbb{D}$ , the point  $g(\zeta)$  is a point of the strong linear convexity of  $G$ .

Moreover, it follows immediately from Lempert's construction that  $G$  may be chosen so that

- $G \subset \mathbb{D} \times \mathbb{C}^{n-1}$ ;
- for any neighborhood  $U \subset \mathbb{C}^{n-1}$  of 0 there is a  $\delta < 1$  such that  $G \setminus (\overline{\mathbb{D}} \times U)$  is contained in  $\delta\mathbb{D} \times \mathbb{C}^{n-1}$ .

Now we shall make use of the following simple observation.

**Lemma 3.2.** *Let  $G$  be a domain contained in  $\mathbb{D} \times \mathbb{C}^{n-1}$  such that  $T := \partial\mathbb{D} \times \{0\}$  is contained in  $\overline{G}$ . Assume moreover that the boundary of  $G$  is smooth and strongly linearly convex in a neighborhood of  $T$ .*

*Then there is a constant  $A > 0$  such that  $|z_1| + A||z'|||^2 < 1$  whenever  $z = (z_1, z') \in G$  is close to  $T$ .*

Postponing the proof of the lemma stated above observe that applying it and making use of properties of  $G$  we see that there is an  $A > 0$  such that

$$(3.1) \quad |z_1|^2 + A|z_j|^2 \leq |z_1| + A||z'|||^2 < 1 \quad \text{for } z = (z_1, z') \in G, j = 2, \dots, n.$$

Then  $G_{A,j}(z) := \frac{z_1}{\sqrt{1-Az_j^2}}$  is a well defined function on  $G$  attaining its values in the unit disc, where  $j = 2, \dots, n$ . It is clear that  $G_{A,j} \circ \Phi$  are left inverses to  $f$  provided that  $A$  is small enough (or  $A = 0$ ).  $\square$

*Proof of Lemma 3.2.* Let  $r$  be a smooth defining function for  $\partial G$  in a neighborhood of  $T$ . The strong linear convexity of  $\partial G$  implies that a function  $z' \mapsto r(z_0, z')$  is strongly convex in a neighborhood of 0 for any  $z_0 \in \mathbb{T}$  (of course  $\{z_0\} \times \mathbb{C}^{n-1}$  is a complex affine space tangent to  $\partial G$  at  $(z_0, 0)$ ). Therefore, there is an open neighborhood  $U \subset \mathbb{C}^{n-1}$  of 0 and there is an  $\alpha > 0$  such that

$$(3.2) \quad r(z_0, z') \geq \alpha||z'|||^2 \quad \text{for any } (z_0, z') \in \mathbb{T} \times U.$$

On the other hand, it follows from the smoothness of  $r$  that

$$(3.3) \quad r(z_1, 0) > C_1(|z_1| - 1) \quad \text{for any } z_1 \in \mathbb{D} \text{ sufficiently close to } \partial\mathbb{D},$$

for some uniform constant  $C_1 > 0$ .

For  $z_1 \in \mathbb{D}$ ,  $z_1 \neq 0$ , let  $\tilde{z}(z_1) := z_1/|z_1|$ . Since  $\frac{\partial r}{\partial z'}(z_0, 0) = 0$  for  $z_0 \in \mathbb{T}$ , making use of the inequalities (3.2) and (3.3) one may easily see that some simple analysis gives constants  $\alpha, \beta > 0$  such that

$$r(z_1, z') \geq -\alpha|z_1 - \tilde{z}(z_1)| + \beta||z'|||^2$$

for any  $z_1 \in \mathbb{D}$  close to  $\partial\mathbb{D}$  and  $z'$  close to 0. From this we immediately deduce that there are positive constants  $A$  and  $C$  such that

$$r(z) \geq C(|z_1| + A||z'|||^2 - 1),$$

whenever  $z$  is sufficiently close to  $T$ . This easily implies the assertion.  $\square$

## 4. UNIQUENESS OF LEFT INVERSES IN CONVEX DOMAINS

**Theorem 4.1.** *Let  $D$  be a convex domain in  $\mathbb{C}^n$  ( $n \geq 2$ ) and let  $f^1, \dots, f^n : \mathbb{D} \rightarrow D$  be complex geodesics such that  $f^1(0) = \dots = f^n(0) = w$  and  $f^1(\sigma) = \dots = f^n(\sigma) = z$  for some  $\sigma \in \mathbb{D} \setminus \{0\}$ . Assume additionally that for some  $\lambda \in \mathbb{D}$  the system of vectors*

$$(4.1) \quad \{f^1(\lambda) - f^n(\lambda), \dots, f^{n-1}(\lambda) - f^n(\lambda)\}$$

*is linearly independent (equivalently, the convex hull of the set  $\{f^1(\lambda), \dots, f^n(\lambda)\}$  is  $(n-1)$ -dimensional). Then there is only one left inverse passing through points  $w$  and  $z$ , i.e. a holomorphic function  $F : D \rightarrow \mathbb{D}$  such that  $F \circ f = \text{id}_{\mathbb{D}}$  for some (equivalently, any) complex geodesic  $f : \mathbb{D} \rightarrow D$  such that  $f(0) = w$ ,  $f(\sigma) = z$ .*

*Proof.* Let  $\emptyset \neq U \subset \subset \mathbb{D}$  be such that for any  $\lambda \in \overline{U}$  the property (4.1) is satisfied. For  $\lambda \in \overline{U}$  define

$$(4.2) \quad g_\lambda(\lambda_1, \dots, \lambda_{n-1}) := \lambda_1 f^1(\lambda) + \dots + \lambda_{n-1} f^{n-1}(\lambda) + (1 - \lambda_1 - \dots - \lambda_{n-1}) f^n(\lambda) = f^n(\lambda) + \lambda_1 (f^1(\lambda) - f^n(\lambda)) + \dots + \lambda_{n-1} (f^{n-1}(\lambda) - f^n(\lambda)).$$

Convexity of  $D$  implies that  $g_\lambda(T) \subset D$ ,  $\lambda \in \overline{U}$ , where  $T := \{t = (t_1, \dots, t_n) : 0 \leq t_j \leq 1, t_1 + \dots + t_{n-1} \leq 1\} \subset \mathbb{C}^{n-1}$ . Let  $T \subset \Omega \subset \mathbb{C}^{n-1}$  be an open connected set such that  $g_\lambda(\Omega) \subset D$ ,  $\lambda \in \overline{U}$ . We easily see that for any  $(t_1, \dots, t_{n-1}) \in T$  the function  $\mathbb{D} \ni \lambda \mapsto g_\lambda(t_1, \dots, t_{n-1}) \in D$  is a complex geodesic in  $D$  for  $(w, z) = (g_0(t_1, \dots, t_{n-1}), g_\sigma(t_1, \dots, t_{n-1}))$ . Let  $F : D \rightarrow \mathbb{D}$  be any left inverse passing through  $w$  and  $z$ . Then,  $F(g_\lambda(t_1, \dots, t_{n-1})) = \lambda$ ,  $\lambda \in \mathbb{D}$ ,  $(t_1, \dots, t_{n-1}) \in T$ . The identity principle for holomorphic functions implies that

$$(4.3) \quad F(g_\lambda(\lambda_1, \dots, \lambda_{n-1})) = \lambda, \lambda \in \overline{U}, (\lambda_1, \dots, \lambda_{n-1}) \in \Omega.$$

Note that the mapping

$$(4.4) \quad \Phi : U \times \Omega \ni (\lambda, \lambda_1, \dots, \lambda_{n-1}) \mapsto g_\lambda(\lambda_1, \dots, \lambda_{n-1}) \in D$$

is holomorphic. We claim that the mapping  $\Phi$  is injective. In fact, the equality  $\Phi(\mu, \mu_1, \dots, \mu_{n-1}) = \Phi(\lambda, \lambda_1, \dots, \lambda_{n-1})$  implies, in view of (4.3), that  $\lambda = \mu$ . Consequently,

$$(4.5) \quad \lambda_1 (f^1(\lambda) - f^n(\lambda)) + \dots + \lambda_{n-1} (f^{n-1}(\lambda) - f^n(\lambda)) = \mu_1 (f^1(\lambda) - f^n(\lambda)) + \dots + \mu_{n-1} (f^{n-1}(\lambda) - f^n(\lambda)).$$

Now the linear independence (property (4.1)) shows that  $\lambda_j = \mu_j$ ,  $j = 1, \dots, n-1$ .

Consequently,  $\Phi(U \times \Omega)$  is a non-empty open subset of  $D$  on which the left inverse  $F$  is uniquely determined. The identity principle for holomorphic functions finishes the proof.  $\square$

**Remark 4.2.** *Note that the assumption of Theorem 4.1 in the case  $n = 2$  means that there are two different complex geodesics passing through  $w$  and  $z$ . In higher dimension we need the existence of 'many more' complex geodesics which would guarantee the uniqueness of left inverses.*

**Remark 4.3.** *Note that the proof Theorem 4.1 relies very much upon the convexity of  $D$ . It is natural to ask the question whether the same result holds without that assumption.*

The above result suggests that the uniqueness of geodesics may imply the non-uniqueness of left inverses. As we shall see from the proposition below this is in general not the case.

**Proposition 4.4.** *Let  $D$  be a pseudoconvex complete Reinhardt domain in  $\mathbb{C}^2$ . Assume that the point  $p_0 := (1, 0)$  lies in the topological boundary of  $D$  and that the boundary of  $D$  is  $C^\infty$  smooth in a neighborhood of  $p_0$ . Let  $f$  denote a complex geodesic  $\mathbb{D} \ni \lambda \mapsto (\lambda, 0) \in D$ .*

*Then  $f$  has a unique left inverse in  $D$  if and only if the boundary of  $D$  is of infinite type at the point  $p_0$ .*

**Remark 4.5.** *It follows from [Din 1989] that in the class of bounded strictly convex domains (i.e. bounded convex domains admitting no segments in the boundary) complex geodesics are always unique. As we know from Theorem 3.1 in the case of strong convexity left inverses are never uniquely determined. This will not be the case if the assumption of strong convexity were relaxed with a geometric assumption of strict convexity. Actually, using Proposition 4.4 we are able to construct a strictly convex and smooth Reinhardt domain such that the only left inverse to  $\mathbb{D} \ni \lambda \mapsto (\lambda, 0) \in D$  is the projection onto the first variable (take for example a smooth Reinhardt domain being additionally strictly convex whose defining function is  $|z| + e^{-|w|^{-2}} - 1$  in a neighborhood of  $p_0$ ).*

*Proof.* Assume first that  $\partial D$  is of finite type at  $p_0$ . Let  $\rho$  be a defining function of  $\partial D$  in a neighborhood of  $p_0$ . Write  $z = te^{i\theta}$ ,  $w = se^{i\eta}$ . We may always assume that

$$\rho(z, w) = \rho(t, s) = t + r(s) - 1$$

for  $(t, s)$  in a neighborhood of  $(1, 0)$ , where  $r(s) = as^k + O(s^{k+1})$ ,  $a > 0$ ,  $k \in \mathbb{N}$ . Therefore, using basic properties of pseudoconvex Reinhardt domains we find that there is  $b > 0$  such  $D \subset \{(z, w) \in \mathbb{C}^2 : |z| + b|w|^k < 1\}$ . Then,

$$F_\beta(z, w) = \frac{z}{1 - \beta w^k}, \quad (z, w) \in D$$

is a left inverse of  $f$  for any  $0 \leq \beta < b$ .

Now assume that  $\partial D$  is of infinite type at  $p_0$  and take any left inverse  $F : D \rightarrow \mathbb{D}$  of  $f$ . Expanding  $F$  in a series and making use of the equality  $F \circ f = \text{id}_{\mathbb{D}}$  we get that

$$F(z, w) = z + zwP_1(z, w) + w^kQ(z, w),$$

for some positive integer  $k$ , holomorphic mappings  $P_1$  and  $Q$  on  $D$  such that either  $Q \equiv 0$  or  $Q(0, 0) \neq 0$ . Observe that the second possibility cannot occur. Actually, suppose that  $Q(0, 0) \neq 0$ . Then, consider the function  $\mathbb{D} \setminus \{0\} \ni \lambda \mapsto F(\lambda^k z, \lambda w) \lambda^{-k}$ , which extends to a well defined holomorphic function on  $\mathbb{D}$  with values in  $\overline{\mathbb{D}}$ . Letting  $\lambda \rightarrow 0$  we get that the mapping  $(z, w) \mapsto z + Q(0, 0)w$  maps  $D$  into the closed unit disc which gives the inclusion  $D \subset \{|z| + |Q(0, 0)||w| < 1\}$  which contradicts the smoothness of the domain.

Therefore,

$$F(z, w) = z(1 + wP_1(z, w)).$$

Our aim is to show that  $P_1 \equiv 0$ . Clearly, this would imply the assertion of the proposition.

Assuming the contrary take  $P$  and a positive integer  $k$  such that  $P_1(z, w) = w^{k-1}P(z, w)$  and  $P(\cdot, 0) \not\equiv 0$ . Take any  $z_0 \in 2^{-1}\mathbb{D}$  such that  $P(z_0, 0) \neq 0$ . Let

$\eta \in \mathbb{R}$  be such that  $|P(z_0, 0)| = e^{i\eta} P(z_0, 0)$ . Considering  $e^{i\eta/k} w$  instead of  $w$  we may assume that  $\eta = 0$ . Let  $A > 0$  be such that  $\operatorname{Re} P(z_0, w) \geq A$  for  $w$  in a neighborhood of 0, say  $|w| < \varepsilon$ , for some  $\varepsilon > 0$ . Certainly  $\operatorname{Re} P(z_0, w) w^k \geq A w^k$  whenever  $0 \leq w < \varepsilon$ .

Making use of the properties of harmonic functions one can see that for any  $2^{-1} < |z| < 1$  and  $|w| < \varepsilon$  there is  $\theta = \theta(z, w) \in \mathbb{R}$  such that  $\operatorname{Re} P(z e^{i\theta}, w) \geq A$ .

Thus, for  $z \in \mathbb{D}$  arbitrarily close to the unit circle and  $0 \leq w < \varepsilon$  we have the following sequence of inequalities:

$$(4.6) \quad |F(e^{i\theta(z,w)} z, w)| = |z| |1 + w^k P(e^{i\theta(z,w)} z, w)| \geq |z| (1 + \operatorname{Re}(w^k P(e^{i\theta(z,w)} z, w))) \geq |z| (1 + A w^k).$$

Making use of (4.6) and the assumptions on  $F$  we infer that the inequality  $1 \geq |z| (1 + A |w|^k)$  holds for  $(z, w) \in \overline{D}$  such that  $|z|$  is close to 1 and  $|w|$  is small. This gives us the contradiction.  $\square$

## 5. LEFT INVERSES IN THE SYMMETRIZED BIDISC (NEVANLINNA-PICK PROBLEM FOR TWO POINTS IN THE SYMMETRIZED BIDISC)

We start this section with recalling a result on a complete description of holomorphic retracts of the diagonal in the bidisc that will be an important tool in the proofs of uniqueness parts of left inverses in both domains: the symmetrized bidisc and the tetrablock. The description is a direct consequence of the following result that we present below in a form that we could apply directly in our paper.

**Theorem 5.1.** (see Example 11.79 in [Agl-McC 2002]) *Let  $F : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{D}$  be a holomorphic function such that  $F(\lambda, \lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ . Then there is an  $h$  lying in the closed unit ball of  $H^\infty(\mathbb{D}^2)$  and  $t \in [0, 1]$  such that*

$$(5.1) \quad F(z_1, z_2) = \frac{t z_1 + (1-t) z_2 - z_1 z_2 h(z_1, z_2)}{1 - ((1-t) z_1 + t z_2) h(z_1, z_2)}, \quad z_1, z_2 \in \mathbb{D}.$$

*Conversely, any function defined as above maps  $\mathbb{D}^2$  to  $\mathbb{D}$  and satisfies the equality  $F(\lambda, \lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ .*

Recall that  $\tilde{k}_{\mathbb{G}_2} = c_{\mathbb{G}_2}$  (see [Agl-You 2004], [Cos 2004]). Consequently, there are complex geodesics passing through arbitrary pair of points. Below we deal with the uniqueness of the left inverse to complex geodesics in the symmetrized bidisc.

To solve this problem it is sufficient to consider two cases that are presented in the theorem giving a complete description of complex geodesics in the symmetrized bidisc that we give below (see [Pfl-Zwo 2005]). Recall that in [Agl-You 2006] another description of complex geodesics in the symmetrized bidisc was given.

**Theorem 5.2.** (see [Pfl-Zwo 2005]). *A holomorphic mapping  $f : \mathbb{D} \rightarrow \mathbb{G}_2$  is a complex geodesic if and only if it is (up to an automorphism of the unit disc and automorphism of the symmetrized bidisc) of one of the following two forms:*

$$(5.2) \quad f(\lambda) = \pi(B(\sqrt{\lambda}), B(-\sqrt{\lambda})),$$

$\lambda \in \mathbb{D}$ , where  $B$  is a non-constant Blaschke product of degree one or two with  $B(0) = 0$ ;

$$(5.3) \quad f(\lambda) = \pi(\lambda, a(\lambda)),$$



where  $a$  is an automorphism of  $\mathbb{D}$  such that  $a(\lambda) \neq \lambda$ ,  $\lambda \in \mathbb{D}$ .

Moreover, the complex geodesics in the symmetrized bidisc are unique (up to automorphisms of the unit disc).

In our considerations (because of the form of automorphisms of  $\mathbb{G}_2$ ) it is sufficient to take in the above theorem

$$(5.4) \quad B(\lambda) = \lambda \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda} \text{ for some } \alpha \in \mathbb{D} \cup \{1\}$$

– we shall assume that below.

Recall that for each complex geodesic one of possible inverses is (up to an automorphism of the unit disc) a function  $\Psi_\omega(s, p) = \frac{2p - \omega s}{2 - \bar{\omega}s}$ , where  $\omega \in \overline{\mathbb{D}}$ . It is elementary to observe that no two such functions are equal up to automorphisms of  $\mathbb{D}$ , i.e. there are no  $\omega_1, \omega_2 \in \overline{\mathbb{D}}$ ,  $a \in \text{Aut } \mathbb{D}$  such that  $\Psi_{\omega_1} \equiv a \circ \Psi_{\omega_2}$ .

Moreover, in all cases except for the case (5.2) and  $\alpha = 1$  the number  $\omega$  is from  $\partial\mathbb{D}$ .

Note that in the case (5.2) and  $\alpha = 0$  any function  $-\bar{\omega}\Psi_\omega$ ,  $\omega \in \partial\mathbb{D}$  is a left inverse to  $f$ .

In the case (5.2) and  $\alpha = 1$  any function  $-\Psi_\omega$ ,  $\omega \in \overline{\mathbb{D}}$  is a left inverse to  $f$ .

Let us also denote the special analytic set lying in  $\mathbb{G}_2$  the so called *royal variety* of  $\mathbb{G}_2$ :

$$(5.5) \quad \mathcal{T} := \{(s, p) \in \mathbb{G}_2 : s^2 = 4p\} = \{(2\lambda, \lambda^2) : \lambda \in \mathbb{D}\}.$$

**Theorem 5.3.** *Let  $f : \mathbb{D} \rightarrow \mathbb{G}_2$  be a complex geodesic of one of the forms from Theorem 5.2.*

*If  $f$  is of the form as in (5.2) with  $B$  satisfying (5.4) then  $f$  has a unique left inverse iff  $\alpha \in \mathbb{D} \setminus \{0\}$ .*

*If  $f$  is of the form as in (5.3) then  $f$  has a unique left inverse iff the function  $a(\lambda) - \lambda$  has two different solutions on the unit circle  $\partial\mathbb{D}$ . Moreover, both cases do hold.*

**Remark 5.4.** *Note that the above theorem is a generalization of Theorem 1.6 in [Agl-You 2006] where the problem of uniqueness of left inverses (but restricted to functions  $\Psi_\omega$ ) is settled.*

*Proof.* In view of the considerations before the theorem in the first case we need to consider only the case (5.2) with  $\alpha \in \mathbb{D} \setminus \{0\}$ .

At first note that there is an (isometric) identification between the space  $H^\infty(\mathbb{G}_2)$  and  $H_s^\infty(\mathbb{D}^2) := \{F \in H^\infty(\mathbb{D}^2) : F(z_1, z_2) = F(z_2, z_1)\}$  given by the formula

$$(5.6) \quad H_s^\infty(\mathbb{D}^2) \ni F \mapsto \{\mathbb{G}_2 \ni (s, p) = \pi(z_1, z_2) \mapsto F(z_1, z_2) \in \mathbb{C}\} \in H^\infty(\mathbb{G}_2).$$

Fix  $\sigma \in \mathbb{D} \setminus \{0\}$ . Consider now the following three point Nevanlinna-Pick problem in the bidisc.

$$(5.7) \quad (0, 0) \rightarrow 0, (B(\sigma), B(-\sigma)) \rightarrow \sigma^2, (B(-\sigma), B(\sigma)) \rightarrow \sigma^2.$$

We already know that there is a holomorphic  $F : \mathbb{G}_2 \rightarrow \mathbb{D}$  such that  $F(f(\lambda)) = \lambda$ . Consequently, the function  $F \circ \pi$  is the Nevanlinna-Pick solution for the above problem. We claim that the solution is unique which would imply the uniqueness of the left inverse to  $f$ .

We make use of the results from [Agl-McC 2002] (Theorem 12.13). According to that result the problem (5.7) will be unique, which shall show that the corresponding

Nevanlinna-Pick problem for the symmetrized bidisc is unique, if the problem (5.7) is

- extremal,
- non-degenerate,
- strictly two-dimensional.

The extremality of the problem is a consequence of the fact that if there were a solution  $\tilde{G}$  of the problem (5.7) with the norm smaller than one then the (symmetrization) function  $G$  given by the formula

$$(5.8) \quad G(z) := (\tilde{G}(z_1, z_2) + \tilde{G}(z_2, z_1))/2$$

would be symmetric with the norm smaller than  $\|G\| < 1$ . This would yield a holomorphic function (denoted by the same letter)  $G$  defined on  $\mathbb{G}_2$  with the norm smaller than one and such that  $G(0) = 0$  and  $G(\pi(B(\sigma), B(-\sigma))) = \sigma^2$ . That would mean that the function  $f$  were not a geodesic - contradiction.

The fact that the problem (5.7) is non-degenerate will follow from the inequality

$$(5.9) \quad |\sigma^2| < \max\{|B(\sigma)|, |B(-\sigma)|\}$$

or equivalently

$$(5.10) \quad |\sigma| < \max\left\{\left|\frac{\sigma - \alpha}{1 - \bar{\alpha}\sigma}\right|, \left|\frac{\sigma + \alpha}{1 - \bar{\alpha}\sigma}\right|\right\}.$$

Note that the Poincaré closed disc with center at  $\sigma$  and the radius  $p(0, \sigma)$  contains at most one of the points:  $\alpha$  and  $-\alpha$  for  $\alpha \in \mathbb{D} \setminus \{0\}$ . This fact implies the last inequality and completes the proof of non-degeneracy.

To show the fact that the problem is strictly two-dimensional let us note that otherwise we would have the existence of a holomorphic  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$  with  $\varphi(0) = 0$ ,  $\varphi(B(\sigma)) = \sigma^2$ ,  $\varphi(B(-\sigma)) = \sigma^2$ . Write  $\varphi(B(\lambda)) = \lambda\psi(\lambda)$  where  $\psi : \mathbb{D} \rightarrow \mathbb{D}$ . Then  $\psi(\sigma) = \sigma$  and  $\psi(-\sigma) = -\sigma$ . Now the Schwarz-Pick Lemma implies that  $\psi = id_{\mathbb{D}}$  and then  $\varphi(B(\lambda)) = \lambda^2$  so

$$(5.11) \quad 0 = \varphi(B(0)) = \varphi(B(\alpha)) = \alpha^2$$

– a contradiction.

Now we consider the case (5.3). Let  $a(\lambda) = \tau \frac{\lambda - \alpha}{1 - \bar{\alpha}\lambda}$  where  $|\tau| = 1$ . Recall that the condition  $a(\lambda) \neq \lambda$ ,  $\lambda \in \mathbb{D}$  is equivalent to the inequality

$$(5.12) \quad |1 - \tau| \leq 2|\alpha|.$$

It follows from the reasoning in [Pfl-Zwo 2005] that  $\Psi_\omega$  is (up to an automorphism) a left inverse iff

$$(5.13) \quad \frac{(1 + \tau\alpha\bar{\omega})^2}{\tau} > 0.$$

It was proven in [Pfl-Zwo 2005] that for the fixed  $\tau$  and  $\alpha$  there is an  $\omega$  satisfying the above inequality iff  $|1 - \tau| \leq 2|\alpha|$ . Therefore, simple geometric observation shows that in the case the inequality in (5.12) is sharp there will be two  $\omega$ 's from  $\partial\mathbb{D}$  for which the inequality in (5.13) holds. This implies that in the case (5.3) with the sharp inequality in (5.12) there is no uniqueness of left inverses.

We are left with the case when the inequality (5.12) becomes equality, i.e. we assume that  $|1 - \tau| = 2|\alpha|$  which is equivalent to the fact that the equation  $a(\lambda) = \lambda$

has a double root on the unit circle. We claim that in this situation there is only one left inverse.

Let  $F$  be a left inverse to  $f$ ,  $f(\lambda) = \pi(\lambda, a(\lambda))$ . Then  $G = F \circ \pi : \mathbb{D}^2 \rightarrow \mathbb{D}$  is a left inverse for the geodesic (in  $\mathbb{D}^2$ ):  $\mathbb{D} \ni \lambda \mapsto (\lambda, a(\lambda)) \in \mathbb{D}^2$ . It is also a symmetric function.

Making use of a description of all solutions of the two-point Nevanlinna-Pick problem

$$(5.14) \quad (0, 0) \rightarrow 0, (1/2, 1/2) \rightarrow 1/2$$

in the bidisc (or equivalently the descriptions of the retracts of the diagonal in the bidisc) we get in view of Theorem 5.1 that

$$(5.15) \quad G(z_1, z_2) = \frac{tz_1 + (1-t)b(z_2) - z_1b(z_2)h(z_1, z_2)}{1 - ((1-t)z_1 + tb(z_2))h(z_1, z_2)}$$

where  $h$  lies in the closed unit ball of  $H^\infty(\mathbb{D}^2)$ ,  $t \in [0, 1]$  and  $b := a^{-1}$ .

Let  $b(\lambda) = \sigma \frac{\lambda - \beta}{1 - \lambda\bar{\beta}}$ , where  $\sigma \in \mathbb{T}$  and  $\beta \in \mathbb{D}_*$ . Note that the relation  $|1 - \sigma| = 2|\beta|$  is satisfied (consequently,  $\sigma$  cannot be equal to  $\pm 1$ ). Observe that  $G(b(\lambda), \lambda) = b(\lambda)$ , in particular, making use of the symmetry of  $G$  we find that  $\lambda \mapsto G(\lambda, b(\lambda))$  vanishes at  $\beta$ . In other words  $t\beta + (1-t)b(0) - \beta b(0)h(\beta, 0) = 0$ . Since  $\beta \neq 0$  we easily find that  $t - (1-t)\sigma + \sigma\beta h(\beta, 0) = 0$ , i.e.

$$(5.16) \quad h(\beta, 0) = \frac{t - (1-t)\sigma}{-\sigma\beta}.$$

The modulus of the term on the right side of the above equation attains its (unique) minimum when  $t = 1/2$ . Therefore,

$$|h(\beta, 0)| \geq \frac{|1 - \sigma|}{2|\beta|} = 1.$$

The maximum principle implies that  $h$  is constant. As we have already shown,  $t$  is uniquely determined ( $t = 1/2$ ) so (the constant function)  $h$  is uniquely determined as well by the equation (5.16).  $\square$

**Remark 5.5.** Note that it follows from the proof of the previous theorem that the uniquely determined function  $G$  in the equality (5.15) is such that  $t = 1/2$  and  $h$  is a constant from  $\mathbb{T}$  (equal to  $\frac{\sigma-1}{2\sigma\beta}$ ).

**Remark 5.6.** Consider a complex geodesic  $\mathbb{D} \ni \lambda \mapsto \pi(a(\lambda), b(\lambda)) \in \mathbb{G}_2$ , where  $a$  and  $b$  are Möbius maps such that the equation  $a(\lambda) = b(\lambda)$ ,  $\lambda \in \mathbb{D}$ , has one double root lying on the unit circle and let  $F : \mathbb{G}_2 \rightarrow \mathbb{D}$  be its left inverse. Recall that the mapping

$$\pi_a : \mathbb{G}_2 \ni \pi(\lambda_1, \lambda_2) \mapsto \pi(a(\lambda_1), a(\lambda_2)) \in \mathbb{G}_2$$

is an automorphism of  $\mathbb{G}_2$  and  $F \circ \pi_a$  is a left inverse of the geodesic of the form  $\lambda \mapsto \pi(\lambda, a^{-1}(b(\lambda)))$ . Therefore, it follows from Remark 5.5 that

$$F(\pi(z_1, z_2)) = \frac{\frac{1}{2}(a^{-1}(z_1) + b^{-1}(z_2)) - a^{-1}(z_1)b^{-1}(z_2)h}{1 - \frac{1}{2}(a^{-1}(z_1) + b^{-1}(z_2))h}, \quad z_1, z_2 \in \mathbb{D},$$

for some  $h \in \mathbb{T}$ .

The facts presented above will be used many times in the next section.

## 6. LEFT INVERSES IN THE TETRABLOCK (NEVANLINNA-PICK PROBLEM FOR TWO POINTS IN THE TETRABLOCK)

Recall that all invariant functions coincide on  $\mathbb{E}$  (see [Edi-Kos-Zwo 2012]). In particular, complex geodesics passing through any pair of points in the tetrablock do exist. Moreover, a special role in the study of the geometry of  $\mathbb{E}$  is played by the set  $\Sigma := \{z \in \mathbb{E} : z_1 z_2 = z_3\}$  called the *royal variety of  $\mathbb{E}$* .

We start with recalling basic properties of  $\mathbb{E}$ . First note that  $\sigma : \mathbb{E} \ni z \mapsto (z_2, z_1, z_3) \in \mathbb{E}$  is an automorphism of  $\mathbb{E}$ .

For  $a, b \in \mathbb{D}$  and  $\omega_1, \omega_2 \in \mathbb{T}$  we put

$$(6.1) \quad \Psi_{a,b,\omega_1,\omega_2}(z_1, z_2, z_3) = \left( \omega_1 \frac{z_1 + a + \bar{b}z_3 + a\bar{b}z_2}{1 + \bar{a}z_1 + \bar{b}z_2 + \bar{a}\bar{b}z_3}, \omega_2 \frac{z_2 + b + \bar{a}z_3 + b\bar{a}z_1}{1 + \bar{a}z_1 + \bar{b}z_2 + \bar{a}\bar{b}z_3}, \omega_1 \omega_2 \frac{z_3 + az_2 + bz_1 + ab}{1 + \bar{a}z_1 + \bar{b}z_2 + \bar{a}\bar{b}z_3} \right),$$

$z = (z_1, z_2, z_3) \in \mathbb{E}$ . Recall that the group of automorphism of  $\mathbb{E}$  is given by

$$\text{Aut}(\mathbb{E}) = \{\Psi_{a,b,\omega_1,\omega_2}, \Psi_{a,b,\omega_1,\omega_2} \circ \sigma : a, b \in \mathbb{D}, \omega_1, \omega_2 \in \mathbb{T}\}$$

(see [You 2008], see also [Kos 2011]).

Similarly as in the case of the symmetrized bidisc for any pair of points there exists a complex geodesic  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  passing through them satisfying one the following three conditions:

- (1)  $\varphi$  lies entirely in the royal variety  $\Sigma$ ,
- (2)  $\varphi$  intersects  $\Sigma$  exactly at one point,
- (3)  $\varphi$  omits  $\Sigma$ .

It is well known (see [You 2008]) that  $\text{Aut}(\mathbb{E})$  acts transitively on  $\Sigma$ . Therefore, if  $\varphi$  is a complex geodesic of  $\mathbb{E}$  such that  $\varphi(\mathbb{D}) \cap \Sigma \neq \emptyset$  we may always assume that  $\varphi(0) = 0$ .

In the case when  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  is a complex geodesic of  $\mathbb{E}$  such that  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  making use of the description of the group of automorphisms of the tetrablock we may assume that  $\varphi(0) = (0, 0, -\beta^2)$  for some  $\beta \in (0, 1)$ .

The following theorem summarizes the description of complex geodesics in the tetrablock obtained in [Abo-Whi-You 2007], [Edi-Zwo 2009] and [Edi-Kos-Zwo 2012].

**Theorem 6.1.** (see [Abo-Whi-You 2007], [Edi-Zwo 2009], [?]) *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  be a complex geodesic in the tetrablock.*

*If  $\varphi(0) = 0$ , then there are  $\omega_1, \omega_2 \in \mathbb{T}$ , a number  $C \in [0, 1]$  and a holomorphic mapping  $\psi : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ ,  $\psi(0) = -C$  such that*

$$(6.2) \quad \varphi(\lambda) = \left( \omega_1 \frac{\psi(\lambda) + C}{1 + C}, \omega_2 \lambda \frac{1 + C\psi(\lambda)}{1 + C}, \omega_1 \omega_2 \lambda \psi(\lambda) \right), \quad \lambda \in \mathbb{D}.$$

*If  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  and  $\varphi(0) = (0, 0, -\beta^2)$  for some  $\beta > 0$ , then there exist  $a, b, c, d \in \overline{\mathbb{D}}$  with  $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$  and  $a\bar{c} + b\bar{d} = 0$  and there exists a holomorphic mapping  $Z : \mathbb{D} \rightarrow \mathbb{D}$  satisfying either  $Z(\lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ , or  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$ , such that*

$$(6.3) \quad \varphi(\lambda) = \left( \frac{A(\lambda)(1 - \beta^2)}{\Delta(\lambda)}, \frac{C(\lambda)(1 - \beta^2)}{\Delta(\lambda)}, \frac{A(\lambda)C(\lambda) - (B(\lambda) + \beta)^2}{\Delta(\lambda)} \right),$$

*where  $A(\lambda) = a^2\lambda + b^2Z(\lambda)$ ,  $B(\lambda) = ac\lambda + bdZ(\lambda)$ ,  $C(\lambda) = c^2\lambda + d^2Z(\lambda)$ , and  $\Delta(\lambda) = (1 + \beta B(\lambda))^2 - A(\lambda)C(\lambda)\beta^2$ .*

Observe that if  $\varphi$  is of the form (6.2), then  $\varphi(\mathbb{D})$  lies entirely in the royal variety  $\Sigma$  if and only if  $C = 0$ . In the other case  $\varphi(\mathbb{D})$  intersects  $\Sigma$  only at 0.

**Remark 6.2.** (see [Edi-Kos-Zwo 2012]). *Suppose that  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  is a complex geodesic of the form (6.3).*

*If  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D}$ , then the condition  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  is equivalent to*

$$(1 + \beta^2)|c||d| \leq \beta.$$

*Moreover, if  $Z(\lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ , then the condition  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  is always satisfied.*

As shown in [Abo-Whi-You 2007] for any complex geodesic  $\varphi$  of  $\mathbb{E}$  intersecting the royal variety there is an  $\omega \in \mathbb{T}$  such that  $\Phi_\omega \circ f \in \text{Aut}(\mathbb{D})$  or  $\Phi_\omega \circ \sigma \circ f \in \text{Aut}(\mathbb{D})$ , where

$$\Phi_\omega(z) = \frac{\omega z_3 - z_1}{\omega z_2 - 1}, \quad z \in \mathbb{E}.$$

On the other hand the family of functions  $\Phi_\omega$  and  $\Phi_\omega \circ \sigma$  is not rich enough to replace in the definition of the Carathéodory distance of  $\mathbb{E}$  the family of all bounded holomorphic functions (see [Edi-Zwo 2009]). In fact, it follows from the reasoning in [Edi-Kos-Zwo 2012] that the functions that we need to add to this family contain the functions  $\tilde{\Phi}_\omega$  and  $\tilde{\Phi}_\omega \circ \sigma$  that we construct below. Namely, it follows from the Rouché Theorem that for any  $z \in \mathbb{E}$ ,  $\omega \in \mathbb{T}$  the equation

$$(6.4) \quad \Phi_\omega(z_1, \lambda z_2, \lambda z_3) = \lambda$$

has exactly one solution  $\lambda =: \tilde{\Phi}_\omega(z)$  in the unit disc. This defines a holomorphic function  $\tilde{\Phi}_\omega : \mathbb{E} \rightarrow \mathbb{D}$  which is a left inverse for some of the geodesics in  $\mathbb{E}$ . After some simple calculations one may find an explicit formula for it:

$$\tilde{\Phi}_\omega(z) := \frac{2z_1}{1 + \omega z_3 + \sqrt{(1 + \omega z_3)^2 - 4\omega z_1 z_2}}.$$

Consider the mapping

$$(6.5) \quad \Pi : \begin{pmatrix} z_1 & a_1 \\ a_2 & z_2 \end{pmatrix} \mapsto (z_1, z_2, z_1 z_2 - a_1 a_2)$$

Recall that the tetrablock may be equivalently given as the image of all (equivalently, all symmetric)  $2 \times 2$  matrices with the operator norm smaller than one under  $\Pi$  (the domain is the Cartan domain of the second type denoted by  $\mathcal{R}_{II}$ ) - see [Abo-Whi-You 2007].

We shall also make use of the connection between the symmetrized bidisc and the tetrablock. Namely, for any  $\omega \in \mathbb{T}$  there is an embedding of  $\mathbb{G}_2$  into  $\mathbb{E}$  given by the formula

$$(6.6) \quad \mathbb{G}_2 \ni (s, p) \mapsto (\omega s/2, s/2, \omega p) \in \mathbb{E}$$

On the other hand for any  $\omega \in \overline{\mathbb{D}}$  we have a mapping (see [Bha 2012], see also [Zwo 2013])

$$(6.7) \quad \mathbb{E} \ni x \mapsto (x_1 + \omega x_2, \omega x_3) \in \mathbb{G}_2.$$

Below we present a (complete) solution of the problem of uniqueness of left inverses in the tetrablock.

**Theorem 6.3.** *Let  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  be a complex geodesic of the form (6.2). Then  $\varphi$  has a unique left inverse if and only if  $C \in (0, 1)$  and the function  $\psi$  is not an automorphism of  $\mathbb{D}$ .*

*If  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  is a complex geodesic of the form (6.3), then  $\varphi$  has a unique left inverse if and only if  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$  and  $|c||d|(1 + \beta^2) = \beta$ .*

The proof of the above theorem is presented below and it is divided into several steps.

**6.1. Geodesics lying entirely in  $\Sigma$ .** Any complex geodesic  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  such that  $\varphi(\mathbb{D}) \subset \Sigma$  and  $\varphi(0) = 0$  is, up to a composition with a linear automorphism of the tetrablock, of the form

$$\varphi(\lambda) = (\lambda, a(\lambda), \lambda a(\lambda)), \quad \lambda \in \mathbb{D},$$

where  $a \in \mathcal{O}(\mathbb{D}, \mathbb{D})$  fixes the origin. Then both  $\Phi_1$  and the projection on the first variable are left inverses for  $\varphi$ .

**6.2. Geodesics intersecting  $\Sigma$  exactly at one point.** Let us focus on the possibility when the complex geodesic  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  satisfies  $\#(\varphi(\mathbb{D}) \cap \Sigma) = 1$  and  $\varphi(0) = 0$ . Then  $\varphi$  is of the form (6.2), where  $C \in (0, 1]$  (recall that  $C = 0$  cannot occur in this case). If  $C = 1$  then  $\varphi(\lambda) = (0, 0, -\omega_1 \omega_2 \lambda)$  and then  $\eta_1 \Psi_1$  and  $\eta \Psi_1 \circ \sigma$  for some  $|\eta_j| = 1$  are left inverses.

Assume now that  $C \in (0, 1)$ . Then  $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ . Rewriting the formula for  $\varphi$  we get that

$$(6.8) \quad \varphi(\lambda) = \left( \omega_1 \frac{(1-C)\psi(\lambda)}{1-C\psi(\lambda)}, \omega_2 \lambda \frac{1-C}{1-C\psi(\lambda)}, \omega_1 \omega_2 \lambda \frac{\psi(\lambda)-C}{1-C\psi(\lambda)} \right), \quad \lambda \in \mathbb{D},$$

where  $C \in (0, 1)$  and  $\psi \in \mathcal{O}(\mathbb{D}, \mathbb{D})$  is such that  $\psi(0) = 0$ . Composing  $\varphi$  with a linear automorphism of  $\mathbb{E}$  we may assume that  $\omega_1 = \omega_2 = 1$ . Write  $\psi(\lambda) = \lambda Z(\lambda)$ , where

- (1) either  $Z$  is a holomorphic self mapping of the unit disc,
- (2) or  $Z(\lambda) = e^{i\theta}$ ,  $\lambda \in \mathbb{D}$ , for some  $\theta \in \mathbb{R}$ .

Let us assume that the first possibility holds, i.e.  $|\psi(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$ . We shall prove that in this case the left inverse is uniquely determined.

Let  $F : \mathbb{E} \rightarrow \mathbb{D}$  be a left inverse of  $\varphi$ . Put

$$\tilde{\varphi}(\lambda, \nu) := \left( \frac{(1-C)\nu}{1-C\nu}, \lambda \frac{1-C}{1-C\nu}, \lambda \frac{\nu-C}{1-C\nu} \right), \quad \lambda, \nu \in \mathbb{D}.$$

Note that  $\tilde{\varphi}(\mathbb{D}^2) \subset \mathbb{E}$ . Observe that  $F \circ \tilde{\varphi}$  does not depend on the second variable. Actually, it suffices to note that  $F \circ \tilde{\varphi}$  is a left inverse for  $\mathbb{D} \ni \lambda \mapsto (\lambda, Z(\lambda)) \in \mathbb{D}^2$  and make use of the uniqueness of left inverses in  $\mathbb{D}^2$  following from the non-uniqueness of geodesics (we may use Theorem 4.1 and remark after that theorem, too). An immediate consequence of this observation is that

$$F(\tilde{\varphi}(\lambda, \omega\lambda)) = \lambda \quad \text{for any } \omega \in \overline{\mathbb{D}}.$$

Let us write  $\tilde{\varphi} = (\tilde{\varphi}_1, \tilde{\varphi}_2, \tilde{\varphi}_3)$ . Note that

$$\tilde{\varphi}_1(\lambda, \omega\lambda) = \omega \tilde{\varphi}_2(\lambda, \omega\lambda), \quad \omega \in \overline{\mathbb{D}}, \lambda \in \overline{\mathbb{D}}.$$

For  $\omega$  in the unit circle considering the embedding

$$\mathbb{G}_2 \ni (s, p) \mapsto (\omega s/2, s/2, \omega p) \in \mathbb{E},$$

one may infer that for any  $\omega \in \mathbb{T}$  the mapping

$$(6.9) \quad f_\omega : \mathbb{D} \ni \lambda \mapsto (2\tilde{\varphi}_2(\lambda, \omega\lambda), \omega^{-1}\tilde{\varphi}_3(\lambda, \omega\lambda)) \in \mathbb{G}_2$$

is a geodesic in the symmetrized bidisc intersecting its royal variety  $\mathcal{T}$  only at  $(0, 0)$ .

Making use of the results of Section 5 (uniqueness of left inverses in the symmetrized bidisc - Theorem 5.3) we find that the (unique) left inverse of  $f_\omega$  is given by the formula

$$(6.10) \quad \mathbb{G}_2 \ni (s, p) \mapsto F(\omega s/2, s/2, \omega p).$$

It follows from the description of left inverses in  $\mathbb{G}_2$  (see Section 5) that

$$(6.11) \quad F(\omega s/2, s/2, \omega p) = b(\omega) \frac{2a(\omega)p - s}{2 - a(\omega)s} \quad (s, p) \in \mathbb{G}_2, \omega \in \mathbb{T},$$

where  $a(\omega), b(\omega) \in \mathbb{T}$ . Putting  $p = 0$  in (6.11) we find that

$$(6.12) \quad (s, \omega) \mapsto \frac{b(\omega)}{2 - a(\omega)s}$$

extends holomorphically to  $\overline{\mathbb{D}}^2$ . Putting  $s = 0$  in (6.12) we find that  $b$  extends holomorphically to  $\overline{\mathbb{D}}$ . Differentiating (6.12)  $k$  times with respect to  $s$  and putting  $s = 0$  we get that  $a^k b$  extends holomorphically to  $\overline{\mathbb{D}}$  for any  $k \in \mathbb{N}$ . From this we easily deduce that  $a$  extends holomorphically to  $\overline{\mathbb{D}}$ .

Since  $F(\omega\tilde{\varphi}_2(\lambda, \omega\lambda), \tilde{\varphi}_2(\lambda, \omega\lambda), \tilde{\varphi}_3(\lambda, \omega\lambda)) = \lambda$  for any  $\lambda \in \mathbb{D}, \omega \in \mathbb{T}$  we find that  $b$  may vanish only at 0 (and then the multiplicity at 0 is at most 1). Therefore, either  $b(w) = e^{i\theta}w$  or  $b(w) = e^{i\theta}$ . Suppose that the first possibility holds. Then

$$F(\omega s/2, s/2, \omega p) = e^{i\theta} \frac{2a(\omega)\omega p - s\omega}{2 - a(\omega)s}, \quad (s, p) \in \mathbb{G}_2, \omega \in \overline{\mathbb{D}}.$$

In particular, putting  $s = 0$  we find that  $F(0, 0, \omega p) = e^{i\theta}a(\omega)\omega p$  for  $\omega, p \in \mathbb{D}$ , whence  $a$  is a unimodular constant and

$$F(z) = e^{i\theta} \frac{az_3 - z_1}{1 - az_2}, \quad z \in \mathbb{E}.$$

Since  $F(\tilde{\varphi}(\lambda, \nu)) = \lambda, \lambda, \nu \in \mathbb{D}$ , putting  $\lambda = 0$  and making use of the above equality we get a contradiction.

Therefore,  $b = e^{i\theta}$ . Putting  $s = 0$  in (6.11) we obtain the equality

$$F(0, 0, \omega p) = e^{i\theta}a(\omega)p,$$

which implies that  $a(\nu) = e^{i\eta}\nu$ . This, together with the equality  $F \circ \tilde{\varphi}(\cdot, \nu) = \text{id}_{\mathbb{D}}$  easily shows that  $e^{i\eta}$  is uniquely defined which shows the uniqueness of  $F$ .

Finally, note that if the case (2) holds, i.e.  $Z(\lambda) = e^{i\theta}, \lambda \in \mathbb{D}$ , then  $\varphi_1(\lambda) = \omega\varphi_2(\lambda), \lambda \in \mathbb{D}$ , for some  $\omega \in \mathbb{T}$ . Thus  $\eta_1\Phi_{\omega_1}$  and  $\eta_2\Phi_{\omega_2} \circ \sigma$  are the left inverses of  $\varphi$ , where  $\eta_i, \omega_i \in \mathbb{T}$  are appropriately chosen.

**6.3. Geodesics omitting  $\Sigma$ .** Let us consider the case when  $\varphi : \mathbb{D} \rightarrow \mathbb{E}$  is a complex geodesic of  $\mathbb{E}$  such that  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  and  $\varphi(0) = (0, 0, -\beta^2)$  for some  $\beta \in (0, 1)$ . Then it follows from Theorem 6.1 that there exist  $a, b, c, d \in \overline{\mathbb{D}}$  with  $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$  and  $a\bar{c} + b\bar{d} = 0$  and there exists a holomorphic mapping  $Z : \mathbb{D} \rightarrow \mathbb{D}$  such that  $Z(\lambda) = \lambda, \lambda \in \mathbb{D}$ , or  $|Z(\lambda)| < |\lambda|, \lambda \in \mathbb{D} \setminus \{0\}$ , such that  $\varphi$  is of the form (6.3).

If  $Z(\lambda) = \lambda$ ,  $\lambda \in \mathbb{D}$ , then  $\varphi_1 = \omega\varphi_2$  for some  $\omega$  in the unit circle. One may check that  $\eta_1\Phi_{\omega_1}$  and  $\eta_2\Phi_{\omega_2} \circ \sigma$  are two different left inverses of  $\varphi$ , for some  $\eta_i, \omega_i \in \mathbb{T}$ ,  $i = 1, 2$ .

If  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$ , then some elementary calculations show that the condition  $\varphi(\mathbb{D}) \cap \Sigma = \emptyset$  means that  $|c||d|(1 + \beta^2) \leq \beta$  (see Remark 6.2). We shall consider two possibilities.

First we focus on the case  $|c||d|(1 + \beta^2) < \beta$ . In this case we show that left inverses are not uniquely determined.

Applying the results of Section 4 of [Edi-Kos-Zwo 2012] we find that there is a Möbius map such that one of the mappings  $(\varphi_1, m\varphi_2, m\varphi_3)$  or  $(m\varphi_1, \varphi_2, m\varphi_3)$  is a complex geodesic of  $\mathbb{E}$  intersecting  $\Sigma$  only at a point  $a \in \mathbb{D}$  for which  $m(a) = 0$ . Losing no generality we suppose that the first possibility holds. Denote  $b := \varphi_1(a)$  and let  $\Psi$  be an automorphism of  $\mathbb{E}$  of the form (see (6.1))

$$\Psi(z) = \Psi_{-b,0,1,1}(z) = \left( \frac{z_1 - b}{1 - \bar{b}z_1}, \frac{z_2 - \bar{b}z_3}{1 - \bar{b}z_1}, \frac{z_3 - \bar{b}z_2}{1 - \bar{b}z_1} \right), \quad z \in \mathbb{E}.$$

Note that  $\Psi$  maps  $(b, 0, 0)$  to 0 and observe that

$$\Psi \circ \tau_m = \tau_m \circ \Psi,$$

where  $\tau_\lambda(z_1, z_2, z_3) = (z_1, \lambda z_2, \lambda z_3)$ ,  $z \in \mathbb{C}^3$ ,  $\lambda \in \mathbb{C}$ . Therefore, composing, if necessary, the geodesic  $\varphi$  and its left inverse with an automorphism of  $\mathbb{E}$  and with a Möbius map we may always reduce the problem to the following one:  $\varphi_1(0) = 0$  and  $\lambda \mapsto \tau_\lambda(\varphi(\lambda)) = (\varphi_1, \lambda\varphi_2, \lambda\varphi_3)$  is a geodesic in  $\mathbb{E}$ . Now one can use the description of the geodesics of  $\mathbb{E}$  intersecting  $\Sigma$  given in (6.8) to find a formula for  $\lambda \mapsto (\varphi_1, \lambda\varphi_2, \lambda\varphi_3)$  which gives the following formula for  $\varphi$

$$\varphi(\lambda) = \left( \omega \frac{(1 - \beta^2)\lambda}{1 - \beta^2\lambda\tilde{Z}(\lambda)}, \omega^{-1} \frac{(1 - \beta^2)\tilde{Z}(\lambda)}{1 - \beta^2\lambda\tilde{Z}(\lambda)}, \frac{\lambda\tilde{Z}(\lambda) - \beta^2}{1 - \beta^2\lambda\tilde{Z}(\lambda)} \right), \quad \lambda \in \mathbb{D},$$

where  $\tilde{Z} : \mathbb{D} \rightarrow \mathbb{D}$  is a holomorphic function or  $\tilde{Z}$  is a unimodular constant and  $\omega \in \mathbb{T}$ . Of course, we may assume that  $\omega = 1$ . Note that the case  $\tilde{Z}(\lambda) = e^{i\theta}$ ,  $\lambda \in \mathbb{D}$ , for some  $\theta \in \mathbb{R}$  is not possible, as it is in a contradiction with the assumption  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$ .

We already know that  $\Phi_1$  is a left inverse for  $\lambda \mapsto (\varphi_1, \lambda\varphi_2, \lambda\varphi_3)$ , i.e.  $\frac{\lambda\varphi_3 - \varphi_1}{\lambda\varphi_2 - 1} = \lambda$ ,  $\lambda \in \mathbb{D}$ . Recall that considering for any  $z \in \mathbb{E}$  the equation

$$\frac{\lambda z_3 - z_1}{\lambda z_2 - 1} = \lambda, \quad \lambda \in \mathbb{D},$$

which has a unique solution, one may define a function  $\tilde{\Phi}_1$  being a left inverse of  $\varphi$  and which can be given explicitly

$$\tilde{\Phi}_1(z) := \frac{2z_1}{1 + z_3 + \sqrt{(1 + z_3)^2 - 4z_1z_2}}.$$

Our aim is to show that there are more left inverses for  $\varphi$ . To do it we shall go to the Cartan symmetric domain of the second type and idea of the proof will rely upon constructing left inverses there. We present below the idea which led us to such a construction. So assume that  $F : \mathbb{E} \rightarrow \mathbb{D}$  is holomorphic and such that  $F \circ \varphi = \text{id}$ .



Repeating the argument used in Subsection 6.2 involving uniqueness of left inverses in the bidisc we deduce that

$$\lambda \mapsto \left( \frac{(1-\beta^2)\lambda}{1-\beta^2\lambda z}, \frac{(1-\beta^2)z}{1-\beta^2\lambda z}, \frac{\lambda z - \beta^2}{1-\beta^2\lambda z} \right)$$

is a complex geodesic in  $\mathbb{E}$  for any  $z \in \mathbb{D}$  and  $F$  is its inverse. In particular,

$$F \left( \frac{(1-\beta^2)\lambda}{1-\beta^2\lambda^2\omega^2}, \omega^2 \frac{(1-\beta^2)\lambda}{1-\beta^2\lambda^2\omega^2}, \frac{\omega^2\lambda^2 - \beta^2}{1-\beta^2\lambda^2\omega^2} \right) = \lambda, \quad \lambda \in \mathbb{D}, \omega \in \mathbb{T}.$$

Therefore,

$$f_\omega : \lambda \mapsto \left( 2 \frac{(1-\beta^2)\lambda}{1-\beta^2\lambda^2\omega^2}, \omega^{-2} \frac{\omega^2\lambda^2 - \beta^2}{1-\beta^2\lambda^2\omega^2} \right)$$

is a complex geodesic in the symmetrized bidisc and  $\mathbb{G}_2 \ni (s, p) \mapsto F(\frac{s}{2}, \omega^2 \frac{s}{2}, \omega^2 p) \in \mathbb{D}$  is its left inverse,  $\omega \in \mathbb{T}$ . Observe that  $f_\omega(\lambda) = \pi(a_\omega(\lambda), b_\omega(\lambda))$ ,  $\lambda \in \mathbb{D}$ , where  $a_\omega(\lambda) = \frac{\lambda - \beta\bar{\omega}}{1 - \lambda\beta\bar{\omega}}$  and  $b_\omega(\lambda) = \frac{\lambda + \beta\bar{\omega}}{1 + \lambda\beta\bar{\omega}}$ ,  $\lambda \in \mathbb{D}$ .

Making use of the description of holomorphic retracts in the bidisc (see Theorem 5.1) we get the formula (note that  $a_\omega^{-1} = b_\omega$ ):

$$(6.13) \quad F \left( \frac{\lambda_1 + \lambda_2}{2}, \omega^2 \frac{\lambda_1 + \lambda_2}{2}, \omega^2 \lambda_1 \lambda_2 \right) = \frac{ta(\lambda_1) + (1-t)b(\lambda_2) - a(\lambda_1)b(\lambda_2)g(\lambda)}{1 - ((1-t)a(\lambda_1) + tb(\lambda_2))g(\lambda)},$$

where  $t = t(\omega) \in [0, 1]$ ,  $a = a_\omega$ ,  $b = b_\omega$  and  $g = g_\omega$  lies in the closed unit ball of  $H^\infty(\mathbb{D}^2)$ . We shall construct a left inverse of the above form with  $t = 1/2$ .

Put  $g_\omega = \omega^2 h_\omega$ ,  $h = h_\omega$ . After some simple calculations we get

$$(6.14) \quad F \left( \frac{\lambda_1 + \lambda_2}{2}, \omega^2 \frac{\lambda_1 + \lambda_2}{2}, \omega^2 \lambda_1 \lambda_2 \right) = \frac{(1-\beta^2)\frac{\lambda_1+\lambda_2}{2} - (\omega^2\lambda_1\lambda_2 + \beta\omega(\lambda_1 - \lambda_2) - \beta^2)h}{1 - \omega\beta(\lambda_1 - \lambda_2) - \beta^2\lambda_1\lambda_2\omega^2 - (1-\beta^2)\frac{\lambda_1+\lambda_2}{2}\omega^2 h}.$$

Observe that for any  $h \in \mathbb{D}$  and  $\omega \in \mathbb{T}$  the mapping in the right side in the formula above lies in the closed unit ball of  $H^\infty(\mathbb{D}^2)$ , as the mapping in the right side of (6.13) lies in the closed unit ball of  $H^\infty(\mathbb{D}^2)$  for any  $g \in \mathbb{D}$  (see Theorem 5.1). Moreover, the denominator  $1 - \omega\beta(\lambda_1 - \lambda_2) - \beta^2\lambda_1\lambda_2\omega^2 - (1-\beta^2)\frac{\lambda_1+\lambda_2}{2}\omega^2 h$  never vanishes if  $\lambda_1, \lambda_2 \in \mathbb{D}^2$ ,  $h \in \mathbb{D}$  and  $\omega \in \mathbb{T}$ . In particular,

$$(6.15) \quad \left| \frac{(1-\beta^2)\frac{\lambda_1+\lambda_2}{2}\omega^2}{1 - \omega\beta(\lambda_1 - \lambda_2) - \beta^2\lambda_1\lambda_2\omega^2} \right| \leq 1$$

providing that  $\lambda_1, \lambda_2 \in \mathbb{D}$  and  $\omega \in \mathbb{T}$ . Note that it is also true for  $\omega \in \mathbb{D}$ . Actually, it suffices to replace  $\lambda_i$  with  $\omega\lambda_i$ ,  $i = 1, 2$ , and make use of the maximum principle.

We shall need the following elementary observation:

**Remark 6.4.** Take a function  $f$  holomorphic on  $\mathcal{R}_{II}$  and continuous on  $\overline{\mathcal{R}_{II}}$ . Let

$$\tilde{\pi} : \mathbb{C}^3 \ni (\lambda_1, \lambda_2, \omega) \mapsto \begin{pmatrix} (\lambda_1 + \lambda_2)/2 & \omega(\lambda_1 - \lambda_2)/2 \\ \omega(\lambda_1 - \lambda_2)/2 & \omega^2(\lambda_1 + \lambda_2)/2 \end{pmatrix} \in \mathcal{M}_{2 \times 2}(\mathbb{C}).$$

Note that  $\Pi(\tilde{\pi}(\lambda_1, \lambda_2, \omega)) = ((\lambda_1 + \lambda_2)/2, \omega^2(\lambda_1 + \lambda_2)/2, \omega^2\lambda_1\lambda_2)$ ,  $\lambda_1, \lambda_2, \omega \in \mathbb{C}$ , where  $\Pi$  is given by the formula (6.5), whence

$$\tilde{\pi}(\mathbb{D}^3) \subset \mathcal{R}_{II}.$$

Since  $f$  attains its maximum on the Shilov boundary of  $\mathcal{R}_{II}$  and the Shilov boundary of  $\mathcal{R}_{II}$  is contained in  $\tilde{\pi}(\mathbb{D}^3)$  (for a description of the Shilov boundary of  $\mathbb{E}$  see [You 2008] or [Kos 2011]) we easily deduce that

$$(6.16) \quad \sup_{\mathcal{R}_{II}} |f| = \sup_{\mathbb{D}^3} |f \circ \tilde{\pi}|.$$

Moreover, if  $f$  is holomorphic on  $\mathcal{R}_{II}$ , then  $f$  is bounded on  $\mathbb{E}$  if and only if  $f \circ \tilde{\pi}$  is bounded on  $\mathbb{D}^3$  (to observe it instead of  $f$  consider its dilatations). Similarly, if  $f$  is additionally bounded, then it satisfies (6.16).

We come back to the proof. The idea is to find a function  $h = h_\omega$  which is analytic on  $\mathcal{R}_{II}$ .

For  $(\lambda_1, \lambda_2, \omega) \in \mathbb{D}^3$  we put  $z_1 := (\lambda_1 + \lambda_2)/2$ ,  $z_2 := \omega^2(\lambda_1 + \lambda_2)/2$  and  $z_3 := \omega^2\lambda_1\lambda_2$ . Clearly  $z = (z_1, z_2, z_3) \in \mathbb{E}$ . Moreover,  $4(z_1z_2 - z_3) = \omega^2(\lambda_1 - \lambda_2)^2$ . This and formula for  $F$  suggest that  $F$  should be considered as the mapping on  $\mathcal{R}_{II}$ .

To do so let us define

$$(6.17) \quad G_h \begin{pmatrix} z_1 & a \\ a & z_2 \end{pmatrix} := \frac{(1 - \beta^2)z_1 - (z_3 + 2\beta a - \beta^2)h}{1 - 2a\beta - \beta^2z_3 - (1 - \beta^2)z_2h} = \frac{(1 - \beta^2)z_1 - (z_1z_2 - (a - \beta)^2)h}{(1 - a\beta)^2 - \beta^2z_1z_2 - (1 - \beta^2)z_2h},$$

where  $z_3 = z_1z_2 - a^2$ .

First observe that if  $h$  lies in the closed unit ball of  $H^\infty(\mathcal{R}_{II})$  then  $G_h$  is well defined on  $\mathcal{R}_{II}$ . Actually, it suffices to show that the denominator appearing in formula (6.17) never vanishes on  $\mathcal{R}_{II}$ . To observe it note that  $(1 - a\beta)^2 - \beta^2z_1z_2 = \det(1 - \beta \begin{pmatrix} a & z_1 \\ z_2 & a \end{pmatrix})$  so this term does not vanish on  $\mathcal{R}_{II}$ , as the spectral norm of the matrix  $\begin{pmatrix} a & z_1 \\ z_2 & a \end{pmatrix}$  is less than 1 on  $\mathcal{R}_{II}$ . Then it suffices to apply Remark 6.4 to

$$\begin{pmatrix} z_1 & a \\ a & z_2 \end{pmatrix} \mapsto \frac{(1 - \beta^2)z_2}{(1 - a\beta)^2 - \beta^2z_1z_2} = \frac{(1 - \beta^2)z_2}{1 - 2a\beta - \beta^2z_3}$$

and make use of the inequality (6.15) to show that the absolute value of the above function is less than 1 on  $\mathcal{R}_{II}$ .

Moreover, applying Remark 6.4 once again we find that  $G_h$  lies in the closed unit ball of  $H^\infty(\mathcal{R}_{II})$ .

Next note that

$$(6.18) \quad G_h \begin{pmatrix} (1 - \beta^2)\lambda & \beta \\ \beta & 0 \end{pmatrix} = \lambda, \quad \lambda \in \mathbb{D},$$

for any  $h$  in the closed unit ball of  $H^\infty(\mathcal{R}_{II})$ .

Finally, some simple computations show that if  $h$  is the left inverse for  $\lambda \mapsto \begin{pmatrix} (1 - \beta^2)\lambda & -\beta \\ -\beta & 0 \end{pmatrix}$  in  $\mathcal{R}_{II}$ , then  $G_h$  is its left inverse, as well, i.e.:

$$(6.19) \quad G_h \begin{pmatrix} (1 - \beta^2)\lambda & -\beta \\ -\beta & 0 \end{pmatrix} = \lambda, \quad \lambda \in \mathbb{D}.$$

Then for such a left inverse  $h$  the following function

$$F_h : \mathbb{E} \ni (z_1, z_2, z_3) \mapsto \frac{1}{2}G_h \left( \begin{array}{cc} z_1 & \sqrt{z_1 z_2 - z_3^2} \\ \sqrt{z_1 z_2 - z_3^2} & z_2 \end{array} \right) + \\ + \frac{1}{2}G_h \left( \begin{array}{cc} z_1 & -\sqrt{z_1 z_2 - z_3^2} \\ -\sqrt{z_1 z_2 - z_3^2} & z_2 \end{array} \right) \in \mathbb{D}$$

is the left inverse for  $\lambda \mapsto ((1 - \beta^2)\lambda, 0, -\beta^2)$  in the tetrablock. Repeating the argument with uniqueness of left inverses in the bidisc we easily deduce that  $F_h$  is the left inverse for the geodesic  $\varphi(\lambda) = \left( \frac{(1-\beta^2)\lambda}{1-\beta^2\lambda\bar{Z}(\lambda)}, \frac{(1-\beta^2)\bar{Z}(\lambda)}{1-\beta^2\lambda\bar{Z}(\lambda)}, \frac{\lambda\bar{Z}(\lambda)-\beta^2}{1-\beta^2\lambda\bar{Z}(\lambda)} \right)$ ,  $\lambda \in \mathbb{D}$ .

So what remains to do is to construct a left inverse  $h$  of  $\lambda \mapsto \begin{pmatrix} (1 - \beta^2)\lambda & -\beta \\ -\beta & 0 \end{pmatrix}$  in  $\mathcal{R}_{II}$  such that

$$(6.20) \quad F_h \not\equiv \tilde{\Phi}_1.$$

We need to introduce some additional notation. Recall that for any  $a \in \mathcal{R}_{II}$  the mapping

$$\varphi_a(x) = (1 - aa^*)^{-\frac{1}{2}}(x - a)(1 - a^*x)^{-1}(1 - a^*a)^{\frac{1}{2}}, \quad x \in \mathcal{R}_{II}$$

is an automorphism of  $\mathcal{R}_{II}$ ,  $\varphi_a(0) = -a$  and  $\varphi_a(a) = 0$ . Let us denote  $\varphi_\beta := \varphi_{B(\beta)}$ , where  $B(\beta) = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$  and  $-1 < \beta < 1$ .

To prove (6.20) note that

$$(6.21) \quad \tilde{\Phi}_1(\lambda, 0, 0) = \lambda, \quad \lambda \in \mathbb{D}.$$

Next, observe that

$$\tilde{h} : \mathcal{R}_{II} \ni \begin{pmatrix} z_1 & a \\ a & z_2 \end{pmatrix} \mapsto z_1 \in \mathbb{D}$$

is a left inverse of  $\mathbb{D} \ni \lambda \mapsto \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{R}_{II}$ . Therefore,

$$h := \tilde{h} \circ \varphi_{-\beta}$$

is a left inverse of  $\lambda \mapsto \begin{pmatrix} (1 - \beta^2)\lambda & -\beta \\ -\beta & 0 \end{pmatrix}$  in  $\mathcal{R}_{II}$ . Thus

$$(6.22) \quad h \left( \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \right) = \tilde{h} \left( \begin{pmatrix} (1 - \beta^2)\lambda & \beta \\ \beta & 0 \end{pmatrix} \right) = (1 - \beta^2)\lambda, \quad \lambda \in \mathbb{D},$$

whence  $F_h(\lambda, 0, 0) = G_h \left( \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \right) = \lambda(1 - \beta^4) \neq \lambda = \tilde{\Phi}_1(\lambda, 0, 0)$  (by (6.21)) so actually,  $\Phi_h$  gives a new left inverse.

We are left with the case when  $\varphi$  is a complex geodesic of the form (6.3), where  $|c||d|(1 + \beta^2) = \beta$  and  $|Z(\lambda)| < |\lambda|$ ,  $\lambda \in \mathbb{D} \setminus \{0\}$ .

Let  $F$  be a left inverse of  $\varphi$ . We shall show that  $F$  is uniquely determined.

As before, one can show that that for any  $Z \in \mathbb{D}$

$$F \left( \frac{A(\lambda, Z)(1 - \beta^2)}{\Delta(\lambda, Z)}, \frac{C(\lambda, Z)(1 - \beta^2)}{\Delta(\lambda, Z)}, \frac{A(\lambda, Z)C(\lambda, Z) - (B(\lambda, Z) + \beta)^2}{\Delta(\lambda, Z)} \right) = \lambda,$$

$\lambda \in \mathbb{D}$ .

Note that either  $|a| < |b|$  or  $|b| < |a|$ . If the first possibility holds we take  $Z = \omega\lambda$ , where  $\omega \in \mathbb{T}$ , and we get that

$$\varphi_\omega : \lambda \mapsto \left( \frac{2A(\lambda, \lambda\omega)(1 - \beta^2)}{\Delta(\lambda, \lambda\omega)}, m(\omega) \frac{A(\lambda, \lambda\omega)C(\lambda, \lambda\omega) - (B(\lambda, \lambda\omega) + \beta)^2}{\Delta(\lambda, \lambda\omega)} \right),$$

where  $m(\omega) = \frac{A(\lambda, \lambda\omega)}{C(\lambda, \lambda\omega)}$ , is a complex geodesic in  $\mathbb{G}_2$  and

$$(s, p) \mapsto F\left(\frac{s}{2}, m(\omega)\frac{s}{2}, m(\omega)p\right)$$

is its left inverse,  $\omega \in \mathbb{T}$ . In the case  $|b| < |a|$  instead of  $\varphi_\omega$  we consider the mapping

$$\psi_\omega : \lambda \mapsto \left( \frac{2C(\lambda, \lambda\omega)(1 - \beta^2)}{\Delta(\lambda, \lambda\omega)}, n(\omega) \frac{A(\lambda, \lambda\omega)C(\lambda, \lambda\omega) - (B(\lambda, \lambda\omega) + \beta)^2}{\Delta(\lambda, \lambda\omega)} \right),$$

where  $n(\omega) = \frac{C(\lambda, \lambda\omega)}{A(\lambda, \lambda\omega)}$ . Therefore, losing no generality we assume that  $|a| < |b|$ .

Replacing  $\lambda$  with  $\eta_1\lambda$ , and  $Z$  with  $\eta_2Z$ , where  $\eta_1, \eta_2 \in \mathbb{T}$  and using the fact that the tetrablock is  $(1, -1, 0)$ -balanced, we may additionally assume that  $a \in \mathbb{R}_{\geq 0}$ ,  $b = c = \sqrt{1 - a^2}$  and  $d = -a$ . Then  $a^2 = \frac{\beta^2}{1 + \beta^2}$ ,  $b^2 = \frac{1}{1 + \beta^2}$  and  $m(\omega) = \frac{\omega + \beta^2}{1 + \beta^2\omega}$ ,  $\omega \in \mathbb{T}$ . Moreover,  $A(\lambda, \lambda\omega) = \frac{\lambda(\beta^2 + \omega)}{1 + \beta^2}$ ,  $B(\lambda, \lambda\omega) = \frac{\lambda\beta(1 - \omega)}{1 + \beta^2}$ ,  $C(\lambda, \lambda\omega) = \frac{\lambda(1 + \omega\beta^2)}{1 + \beta^2}$ ,  $\Delta(\lambda, \lambda\omega) = 1 + \lambda \frac{2\beta^2(1 - \omega)}{1 + \beta^2} - \lambda^2\beta^2\omega$ . Consequently,

$$(6.23) \quad (\varphi_\omega(\lambda))_1 = \frac{\lambda 2 \frac{(\beta^2 + \omega)(1 - \beta^2)}{1 + \beta^2}}{1 + \lambda \frac{2\beta^2(1 - \omega)}{1 + \beta^2} - \lambda^2\beta^2\omega}$$

and

$$(6.24) \quad (\varphi_\omega(\lambda))_2 = \frac{\frac{\lambda^2(\beta^2 + \omega)(1 + \omega\beta^2) - (\lambda\beta(1 - \omega) + \beta(1 + \beta^2))^2}{(1 + \beta^2)^2}}{1 + \lambda \frac{2\beta^2(1 - \omega)}{1 + \beta^2} - \lambda^2\beta^2\omega}.$$

Using the description of geodesics in the symmetrized bidisc omitting its royal variety (compare Theorem 5.2) we find that there are Möbius maps  $c_\omega$  and  $d_\omega$  such that  $\varphi_\omega = \pi(c_\omega, d_\omega)$ ,  $\omega \in \mathbb{T}$ . In particular, making use of the description of holomorphic retracts in the bidisc we deduce that

$$(6.25) \quad F\left(\frac{\lambda_1 + \lambda_2}{2}, \frac{1}{m(\omega)} \frac{\lambda_1 + \lambda_2}{2}, \frac{1}{m(\omega)} \lambda_1 \lambda_2\right) = \frac{t_\omega c_\omega^{-1}(\lambda_1) + (1 - t_\omega) d_\omega^{-1}(\lambda_2) - c_\omega^{-1}(\lambda_1) d_\omega^{-1}(\lambda_2) h_\omega(\lambda_1, \lambda_2)}{1 - ((1 - t_\omega) c_\omega^{-1}(\lambda_1) + t_\omega d_\omega^{-1}(\lambda_2)) h_\omega(\lambda_1, \lambda_2)},$$

$\lambda_1, \lambda_2 \in \mathbb{D}$ ,  $\omega \in \mathbb{T}$ , for some  $h_\omega \in H^\infty(\mathbb{D}^2)$  such that  $\|h_\omega\| \leq 1$  and  $t_\omega \in [0, 1]$ .

The first crucial observation is that  $\pi(c_{-1}, d_{-1})$  is a complex geodesic in  $\mathbb{G}_2$  having one left inverse. More precisely, the equation  $c_{-1} = d_{-1}$  has only one solution (which lies in the unit circle) and then we apply Theorem 5.3. To prove it let us observe that

$$\varphi_{-1}(\lambda) = \left( \frac{-\frac{(1 - \beta^2)^2}{1 + \beta^2} \lambda}{1 + \frac{4\beta^2}{1 + \beta^2} \lambda + \beta^2 \lambda^2}, \frac{\lambda^2 + \frac{4\beta^2}{1 + \beta^2} \lambda + \beta^2}{1 + \frac{4\beta^2}{1 + \beta^2} \lambda + \beta^2 \lambda^2} \right), \quad \lambda \in \mathbb{D}.$$

Let  $a := \frac{-2\beta^2}{1+\beta^2} - i\beta\frac{1-\beta^2}{1+\beta^2} \in \mathbb{D}$  be a solution of the equation  $\lambda^2 + \frac{4\beta^2}{1+\beta^2}\lambda + \beta^2 = 0$ . Then it is easy to check that

$$c_{-1}(\lambda) = i\frac{|a|}{a}\frac{\lambda + a}{1 + \bar{a}\lambda} \quad \text{and} \quad d_{-1}(\lambda) = -i\frac{a}{|a|}\frac{\lambda + \bar{a}}{1 + a\lambda}, \quad \lambda \in \mathbb{D}.$$

A direct computation allows us to observe that the equation  $c_{-1}(\lambda) = d_{-1}(\lambda)$  has one double root  $\lambda = 1$ . So actually, the left inverse of  $\pi(c_{-1}, d_{-1})$  is uniquely defined. Moreover, it follows from Remark 5.6 that

$$(6.26) \quad t_{-1} = \frac{1}{2}$$

and  $h_{-1}$  is a unimodular constant. To compute its value note that  $(c_{-1})^{-1}(\lambda) = -i\frac{a}{|a|}\frac{\lambda - i|a|}{1 + i|a|\lambda}$  and  $(d_{-1})^{-1}(\lambda) = i\frac{|a|}{a}\frac{\lambda + i|a|}{1 - i|a|\lambda}$ . Making use of Remark 5.6 again we deduce after some calculations that  $h_{-1} = -1$ . Putting in (6.25)  $\lambda_1 = \lambda_2 = 0$  we find that

$$(6.27) \quad F(0) = -\beta^2.$$

Differentiating (6.25) at the point  $(c_\omega(0), d_\omega(0))$  with respect to  $\lambda_1$  and  $\lambda_2$  respectively we get that

$$\begin{aligned} t_\omega(c_\omega^{-1})'(c_\omega(0)) &= \frac{1}{2}\frac{\partial F}{\partial x_1}(0, 0, -\beta^2) + \\ &\quad \frac{1}{m(\omega)}\frac{1}{2}\frac{\partial F}{\partial x_2}(0, 0, -\beta^2) + \frac{1}{m(\omega)}d_\omega(0)\frac{1}{2}\frac{\partial F}{\partial x_3}(0, 0, -\beta^2) \end{aligned}$$

and

$$\begin{aligned} (1 - t_\omega)(d_\omega^{-1})'(d_\omega(0)) &= \frac{1}{2}\frac{\partial F}{\partial x_1}(0, 0, -\beta^2) + \\ &\quad \frac{1}{m(\omega)}\frac{1}{2}\frac{\partial F}{\partial x_2}(0, 0, -\beta^2) + \frac{1}{m(\omega)}c_\omega(0)\frac{1}{2}\frac{\partial F}{\partial x_3}(0, 0, -\beta^2). \end{aligned}$$

As  $c_\omega(0) + d_\omega(0) = 0$  adding the above equalities we find that

$$(6.28) \quad \frac{t_\omega d'_\omega(0) + (1 - t_\omega)c'_\omega(0)}{c'_\omega(0)d'_\omega(0)} = \frac{\partial F}{\partial x_1}(0, 0, -\beta^2) + \frac{1}{m(\omega)}\frac{\partial F}{\partial x_2}(0, 0, -\beta^2).$$

Differentiating the components of the equality  $\pi(c_\omega, d_\omega) = \varphi_\omega$  we get that

$$c'_\omega(0) + d'_\omega(0) = \frac{2(1 - \beta^2)}{1 + \beta^2}(\beta^2 + \omega)$$

and (recall  $c_\omega(0) + d_\omega(0) = 0$  and  $c_\omega(0)d_\omega(0) = -m(\omega)\beta^2$ ,  $\omega \in \mathbb{T}$ )

$$c'_\omega(0) - d'_\omega(0) = \pm \frac{2\beta}{1 + \beta^2}(1 - \omega)\sqrt{m(\omega)}(1 - \beta^2),$$

for some branch of the square root  $\sqrt{m(\omega)}$ . The equalities above simply imply that

$$c'_\omega(0)d'_\omega(0) = \left(\frac{1 - \beta^2}{1 + \beta^2}\right) ((\beta^2 + \omega)^2 - \beta^2(1 - \omega)^2 m(\omega)).$$

Let us denote  $A := \frac{\partial F}{\partial x_1}(0, 0, -\beta^2)$ ,  $B := \frac{\partial F}{\partial x_2}(0, 0, -\beta^2)$ . Rewriting (6.28) we get that

$$\frac{(t_\omega - \frac{1}{2})(d'_\omega(0) - c'_\omega(0)) + \frac{1}{2}(c'_\omega(0) + d'_\omega(0))}{c'_\omega(0)d'_\omega(0)} = A + \frac{1}{m(\omega)}B.$$

Some simple computations lead to

$$(6.29) \quad \pm 2(t_\omega - \frac{1}{2})\sqrt{m(\omega)}\frac{1-\beta^2}{1+\beta^2}\beta(1-\omega) = -\frac{1-\beta^2}{1+\beta^2}(\beta^2 + \omega) + \left(\frac{1-\beta^2}{1+\beta^2}\right)^2 ((\beta^2 + \omega)^2 - \beta^2 m(\omega)(1-\omega)^2) \left(A + \frac{1}{m(\omega)}B\right).$$

Putting here  $\omega = -1$  and making use of (6.26) we find that

$$A - B = \frac{-1}{1+\beta^2}.$$

On the other hand putting in (6.29)  $\omega = 1$  we get

$$A + B = \frac{1}{1-\beta^2}.$$

Looking at (6.29) carefully we see that there are holomorphic mappings  $\varphi$  and  $\psi$  in a neighborhood of  $\overline{\mathbb{D}}$  (which may be given explicitly) such that

$$\pm(t_\omega - \frac{1}{2})\sqrt{m(\omega)} = \varphi(\omega)m(\omega) + \psi(\omega), \quad \omega \in \mathbb{T}.$$

Raising the equation above to the second power we get

$$(6.30) \quad (t_\omega - \frac{1}{2})^2 m(\omega) = (\varphi(\omega)m(\omega) + \psi(\omega))^2, \quad \omega \in \mathbb{T}.$$

Using the fact that  $t_\omega$  attains only real values we easily find that there are  $\alpha' \in \mathbb{C}$  and  $\alpha'' \in \mathbb{R}$  such that  $(t_\omega - \frac{1}{2})^2 = \overline{\alpha'}m(\omega)^{-1} + \alpha'' + \alpha'm(\omega)$  (this observation appears in the literature as the Gentili's lemma, see [Gen 1987]). Since  $t_{-1} = 1/2$  we get that  $\alpha'' = 2 \operatorname{Re} \alpha'$ . Comparing multiplicities of roots we deduce that  $\alpha' \in \mathbb{R}$  and  $\alpha'' = 2\alpha'$ .

Concluding, we have shown the existence of  $\alpha$  such that  $\pm(t_\omega - \frac{1}{2})\sqrt{m(\omega)} = \alpha(m(\omega) + 1)$ ,  $\omega \in \mathbb{T}$ . Note that  $\alpha = \psi(-\beta^2)$  and the last value is equal to  $\frac{\beta}{2(1+\beta^2)}$ .

Let  $\gamma_\omega$  and  $\delta_\omega$  be that  $c_\omega(\gamma_\omega) = d_\omega(\delta_\omega) = 0$ . Our aim is to make use of the formula (6.25) with  $\lambda_1 = \lambda_2 = 0$ . Note that this formula may be rewritten as (remember about (6.27)):

$$(6.31) \quad \frac{\frac{1}{2}(\delta_\omega^{-1} + \gamma_\omega^{-1}) + (t_\omega - \frac{1}{2})(\delta_\omega^{-1} - \gamma_\omega^{-1}) - h_\omega(0)}{\gamma_\omega^{-1}\delta_\omega^{-1} - (\frac{1}{2}(\delta_\omega^{-1} + \gamma_\omega^{-1}) - (t_\omega - \frac{1}{2})(\delta_\omega^{-1} - \gamma_\omega^{-1}))h_\omega(0)} = -\beta^2.$$

Recall that  $\Delta(\lambda, \omega\lambda) = 1 + 2\frac{\beta^2}{1+\beta^2}(1-\omega)\lambda - \lambda^2\beta^2\omega$ . Certainly,  $\frac{1}{\gamma_\omega}$  and  $\frac{1}{\delta_\omega}$  solve the equation

$$\Delta(\lambda, \omega\lambda) = 0.$$

From this we easily find that

$$\left(t_\omega - \frac{1}{2}\right)^2 (\gamma_\omega^{-1} - \delta_\omega^{-1})^2 = \left(\frac{1+\omega}{1+\beta^2}\right)^2.$$

Therefore  $(t_\omega - \frac{1}{2})(\gamma_\omega^{-1} - \delta_\omega^{-1}) = s_\omega$ , where either  $s_\omega = \frac{1+\omega}{1+\beta^2}$  or  $s_\omega = -\frac{1+\omega}{1+\beta^2}$ . Putting it in (6.31) we find that

$$h_\omega(0) \left(1 + \beta^2 \frac{\omega - 1}{1 + \beta^2} - \beta^2 s_\omega\right) = \frac{\omega - 1}{1 + \beta^2} - \omega + s_\omega.$$

If  $s_\omega = -\frac{1+\omega}{1+\beta^2}$  we get that  $|h_\omega(0)| > 1$ ,  $\omega \in \mathbb{T} \setminus \{-1\}$ , which is impossible. Therefore  $s_\omega = \frac{1+\omega}{1+\beta^2}$  and  $h_\omega(0) = \omega$ ,  $\omega \in \mathbb{T}$ . The maximum principle implies that  $h_\omega$  is constant,  $\omega \in \mathbb{T}$ , and the identity principle implies that  $F$  is uniquely determined (note that  $t_\omega$  is also uniquely determined).

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